The envelope theory and the improved envelope theory: an overview of these approximation methods

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The ET for systems of all identical particles

What is the ET ?

The ET provides approximation of spectrum for a very large class of Hamiltonians [1].

[1] Semay, Ducobu (2016) Eur. J. Phys., **37**, 045403

[2] Silvestre-Brac, Semay, Buisseret (2012) J. Phys. Math., **4**, 120601

The ET for systems of all identical particles

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For today, we will use:

$$H = \sum_{i=1}^{N} T(p_i) + \sum_{i<j=2}^{N} V(r_{ij})$$

with $p_i = |\vec{p}_i|$ and $r_{ij} = |\vec{r}_i - \vec{r}_j|$.

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The basic idea of the ET is to approximate this hamiltonian with a set of **harmonic oscillator (HO) Hamiltonian** [2].

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What is the ET ?

Remark about the N identical body HO:

$$H_{OH} = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m} + \sum_{i<j=2}^{N} \rho \vec{r}_{ij}^{\,2} - \frac{\vec{P}^2}{2Nm} \quad \text{with } \vec{P} = \sum_{i=1}^{N} \vec{p}_i.$$

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The exact spectrum for this Hamiltonian can be analytically found [3]:

$$E_{n_i,l_i} = \sqrt{\frac{2N\rho}{m}} Q(N) \text{ with } Q(N) = \sum_{i=1}^{N-1} (2n_i + l_i + D/2)$$

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(We use natural units).

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The ET for systems of all identical particles

Essential idea behind the ET

Aim: find the spectrum of this generic Hamiltonian

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To this end, we introduce an **auxiliary Hamiltonian**:

$$\tilde{H} = \sum_{i=1}^{N} \tilde{T}_i(p_i, \mu_i) + \sum_{i<j=2}^{N} \tilde{V}_{ij}(r_{ij}, \rho_{ij}).$$

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- μ_i and ρ_{ij} are called **auxiliary fields** [2]. For now, they depend on the variables $\vec{p}_i$ and $\vec{r}_i$.

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- Illustration for the potential:

$$\tilde{V}_{ij}(r_{ij}, \rho_{ij}) = \rho_{ij} r_{ij}^2 + V(J(\rho_{ij})) + \rho_{ij} J(\rho_{ij})^2$$

where $J(x)$ is the inverse of $V'(x)/(2x)$.

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$\Rightarrow$ $\tilde{H}$ looks like an HO Hamiltonian.

- Setting the constraints $\frac{\delta \tilde{H}}{\delta \mu_i} = \frac{\delta \tilde{H}}{\delta \rho_{ij}} = 0$, H is recovered [2].

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Idea behind the ET: to replace the **auxiliary fields** by **auxiliary parameters** [2]. $\tilde{H}$ becomes then an HO Hamiltonian.

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The spectrum of H is approximately recovered by setting $\left.\frac{\partial \tilde{E}}{\partial \mu_i}\right|_{\mu_i=\mu_{i0}} = \left.\frac{\partial \tilde{E}}{\partial \rho_{ij}}\right|_{\rho_{ij}=\rho_{ij0}} = 0$ where $\tilde{E}$ is an energy level of $\tilde{H}$ [2].

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Conclusion: for each energy level, the method gives a set $\{\mu_{i0}, \rho_{ij0}\}$ that approximates H.

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⇒ Basically H **is approximated by a set of auxiliary HO Hamiltonians**, one for each level [2].

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$\Rightarrow$ Basically H **is approximated by a set of auxiliary HO Hamiltonians**, one for each level [2].

It can be shown that the constraints

$$\left.\frac{\partial \tilde{E}}{\partial \mu_i}\right|_{\mu_i=\mu_{i0}} = \left.\frac{\partial \tilde{E}}{\partial \rho_{ij}}\right|_{\rho_{ij}=\rho_{ij0}} = 0$$

lead to a set of three equations called **compact equations of the ET** [4].

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The ET for systems of all identical particles

Compact equations

Practical user necessary informations: let us take the following generic Hamiltonian

$$H = \sum_{i=1}^{N} T(|\vec{p}_i|) + \sum_{i<j=2}^{N} V(|\vec{r}_i - \vec{r}_j|),$$

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the next system gives an approximation for its spectrum [4]:

$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ Np_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \\ \sqrt{C_N^2}\rho_0 p_0 = Q \end{cases}$$

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- with $Q(N) = \sum_{i=1}^{N-1} (2n_i + l_i) + (N-1)\frac{D}{2}$,

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- with $Q(N) = \sum_{i=1}^{N-1} (2n_i + l_i) + (N-1)\frac{D}{2}$,
- with, $\forall\ i, j$, $p_0^2 = \langle \vec{p}_i^{\,2} \rangle$ and $\rho_0^2 = \langle (\vec{r}_i - \vec{r}_j)^2 \rangle$.

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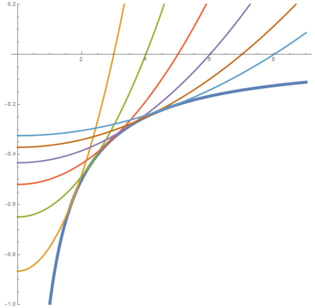
The ET for systems of all identical particles

Commentary about ET

- **Variational character**: ET can give an upper or a lower bound [2].



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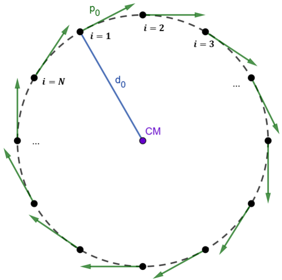

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- The compact equations have a nice **semi-classical interpretation** [4].
- Solution may be **analytical** with N as a variable [5].

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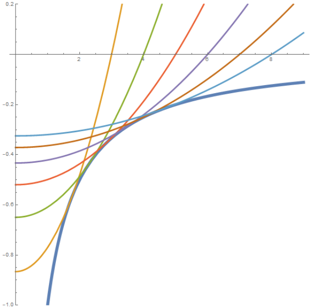

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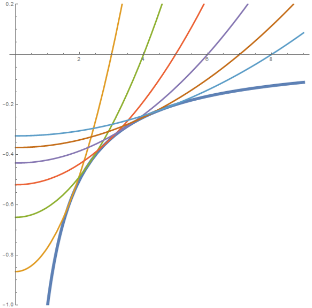

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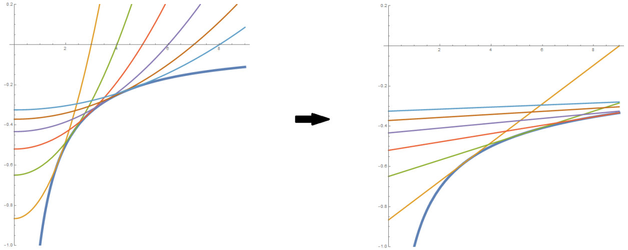

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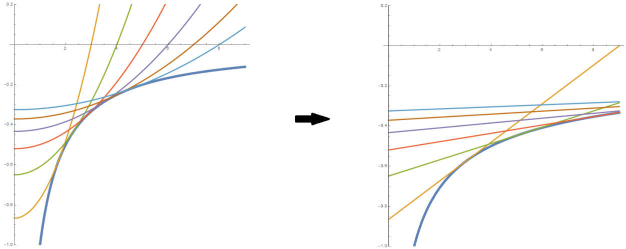

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    - if it is not the case $\Rightarrow$ no defined variational character

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A semi-classical interpretation

- **Semi-classical interpretation** for the compact equations [4]:

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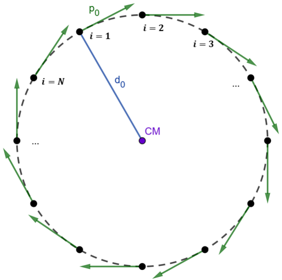

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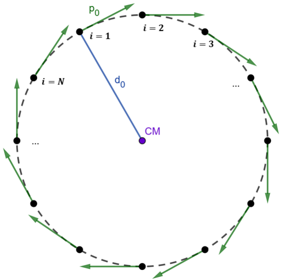

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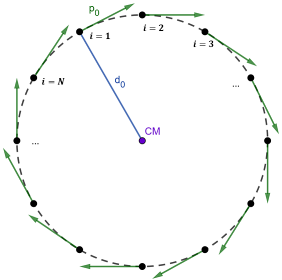

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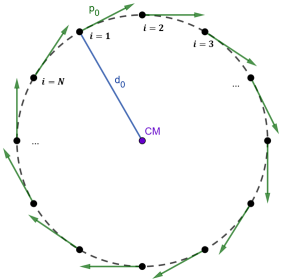

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$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ \sqrt{C_N^2}\rho_0 p_0 = Q \\ Np_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \end{cases}$$

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Analycity

Hamiltonian: $T(p) = \frac{\vec{p}^{\,2}}{2m}$ $V(x) = ax^2$

$\Rightarrow$ "Approximated" spectrum: $E = Q\sqrt{\frac{2N}{m}a}$

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

The ET for systems of all identical particles

Analycity

Hamiltonian: $T(p) = \frac{\vec{p}^{\,2}}{2m}$ $V(x) = ax^2$

$\Rightarrow$ "Approximated" spectrum: $E = Q\sqrt{\frac{2N}{m}a}$

Hamiltonian: $T(x) = Fx^\alpha$ $V(x) = \mathrm{sgn}(\beta)Gx^\beta$

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$$E = \mathrm{sgn}(\beta)(\beta+\alpha)\left(\left(\frac{NF}{|\beta|}\right)^\beta\left(\frac{G}{\alpha}\right)^\alpha\left(\sqrt{C_N^2}\right)^{2\alpha-\alpha\beta}Q^{\alpha\beta}\right)^{1/(\alpha+\beta)}$$

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

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Hamiltonian: $T(p) = \frac{\vec{p}^{\,2}}{2m}$ $V(x) = -V_g e^{-\frac{(x_i-x_j)^2}{a^2}}$

$\Rightarrow$ Approximated spectrum : $E = -C_N^2 V_g e^{2W(\delta)}(2W(\delta)+1)$ where $\delta = -\frac{1}{2}\left(\frac{N}{V_g 2m(C_N^2)^2 a^2} Q^2\right)^{1/2}$ and $W(x)$ is a Lambert function.

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

The ET for systems of all identical particles

Tests

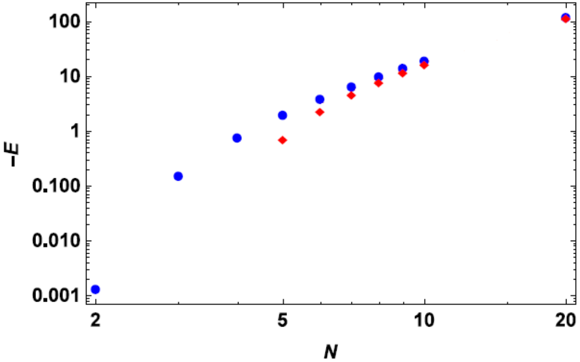

Figure: Biding energy for weakly-interacting bosons (gaussian interaction) with $D = 3$ - Exact results in circles, ET results in diamonds.

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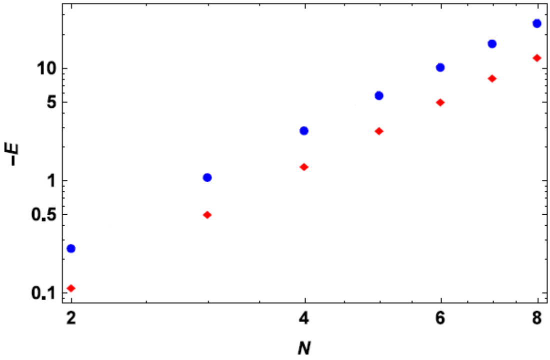

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The ET for systems of all identical particles

To improve the ET

- **The improved ET**: Q has a strong degeneracy.

[7] Lobashev, Trunov (2009) J. Phys. A, **42**, 345202
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[8] Semay (2015) Eur. Phys. J. Plus, **130**, 156

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$$Q = \phi \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \sum_{i=1}^{N-1} l_i + (N-1)\frac{D-2}{2}.$$

[7] Lobashev, Trunov (2009) J. Phys. A, **42**, 345202

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[7] Lobashev, Trunov (2009) J. Phys. A, **42**, 345202

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[8] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

What is the DOSM ?

Strategy behind the DOSM [8]:

(1) To start with a classical purely orbital solution

[8] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

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Strategy behind the DOSM [8]:

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(2) To start a radial perturbation (still classically)

[8] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

What is the DOSM ?

Strategy behind the DOSM [8]:

(1) To start with a classical purely orbital solution

(2) To start a radial perturbation (still classically)

(3) To quantify the perturbation

[8] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The ET for systems of all identical particles

To apply the DOSM to the compact equations

Strategy:

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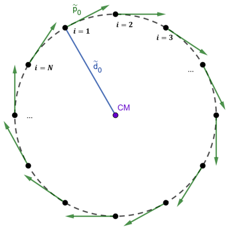
$$\begin{cases} \tilde{E}_0 = NT(\tilde{p}_0) + C_N^2 V(\tilde{\rho}_0) \\ \sqrt{C_N^2}\tilde{\rho}_0\tilde{p}_0 = \lambda \\ N\tilde{p}_0 T'(\tilde{p}_0) = C_N^2\tilde{\rho}_0 V'(\tilde{\rho}_0) \end{cases}$$



The ET for systems of all identical particles

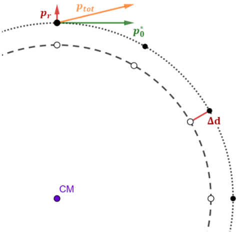
To apply the DOSM to the compact equations

Strategy:

(1) To start with a classical purely orbital solution
(2) To start a radial perturbation (still classically)
(3) To quantify the perturbation

$$\tilde{\rho}_0 \xrightarrow{pert.} \tilde{\rho}_0 + \Delta\rho$$

$$\tilde{p}_0 \xrightarrow{pert.} \sqrt{p_r^2 + \left(\frac{\lambda}{\sqrt{C_N^2}(\tilde{\rho}_0 + \Delta\rho)}\right)^2}$$



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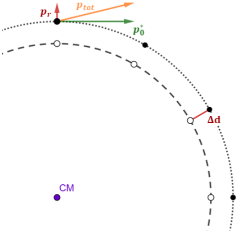
$$\tilde{\rho}_0 \xrightarrow{pert.} \tilde{\rho}_0 + \Delta\rho$$

$$\tilde{p}_0 \xrightarrow{pert.} \tilde{p}_0 \left(1 + \frac{p_r^2}{2\tilde{p}_0{}^2} - \frac{\Delta\rho}{\tilde{\rho}_0} + \frac{\Delta\rho^2}{\tilde{\rho}_0{}^2}\right)$$



The ET for systems of all identical particles

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Strategy:

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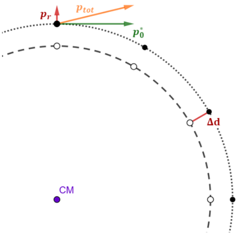
$$\tilde{E}_0 = NT(\tilde{p}_0) + C_N^2 V(\tilde{\rho}_0) \xrightarrow{pert.}$$

$$\Delta E = \left(\frac{N}{2\tilde{p}_0} T'(\tilde{p}_0)\right) p_r^2 + \left(\frac{N\tilde{p}_0}{\tilde{\rho}_0{}^2} T'(\tilde{p}_0) + \frac{N\tilde{p}_0{}^2}{2\tilde{\rho}_0{}^2} T''(\tilde{p}_0) + \frac{C_N^2}{2} V''(\tilde{\rho}_0)\right) \Delta\rho^2$$



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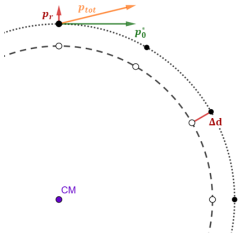
$$\tilde{E}_0 = NT(\tilde{\rho}_0) + C_N^2 V(\tilde{\rho}_0) \xrightarrow{pert.}$$

$$\Delta E = \frac{p_r^2}{2\mu} + \frac{k}{2}\Delta\rho^2$$



The ET for systems of all identical particles

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$$\Delta E = \frac{p_r^2}{2\mu} + \frac{k}{2}\Delta\rho^2 \;\rightarrow\; \Delta E = \sqrt{\frac{k}{\mu}}\left(n+\frac{1}{2}\right)$$

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Test on HO $\rightarrow \sum_{i=1}^{N-1}\left(n_i + \frac{1}{2}\right) = \sqrt{C_N^2}\left(n + \frac{1}{2}\right)$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

The DOSM for systems of all identical particles

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(1) To restructure Q,

$$Q = \phi \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \lambda$$

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$$= \lambda \epsilon + \lambda \qquad \text{avec } \epsilon = \frac{\phi}{\lambda} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right)$$

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To apply the DOSM to the compact equations

Aim: to compare ET and DOSM in the same conditions.

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(2) To develop for $\epsilon \ll 1$ (radial perturbation),

$$Q = \lambda\epsilon + \lambda$$
$$\rho_0 = \tilde{\rho}_0 + \Delta\rho$$
$$p_0 = \tilde{p}_0 + \Delta p$$

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The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

(3) Determination of ϕ

$$\text{DOSM} \rightarrow \Delta E = \sqrt{\frac{k}{C_N^2 \mu}} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right)$$

$$\text{improved ET} \rightarrow \Delta E = N \tilde{p}_0 \, T'(\tilde{p}_0) \frac{\phi}{\lambda} \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right)$$

The DOSM for systems of all identical particles

To apply the DOSM to the compact equations

(3) Determination of ϕ

$$\phi = \frac{\lambda}{N\tilde{p}_0 T'(\tilde{p}_0)} \sqrt{\frac{k}{C_N^2 \mu}}$$

$$\text{with } \mu = \frac{\tilde{p}_0}{NT'(\tilde{p}_0)}$$

$$\text{and } k = \frac{2N\tilde{p}_0}{\tilde{\rho}_0{}^2} T'(\tilde{p}_0) + \frac{N\tilde{p}_0{}^2}{\tilde{\rho}_0{}^2} T''(\tilde{p}_0) + C_N^2 V''(\tilde{\rho}_0)$$

The DOSM for systems of all identical particles

Methodology

Methodology [8]:

[8] Semay (2015) Eur. Phys. J. Plus, **130**, 156

The DOSM for systems of all identical particles

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Methodology [8]:

- To choose l_i and calculate λ,

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- To choose l_i and calculate λ,
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$$\phi = \frac{\lambda}{N\tilde{p}_0 T'(\tilde{p}_0)}\sqrt{\frac{k}{C_N^2 \mu}} \text{ with } \mu = \frac{\tilde{p}_0}{NT'(\tilde{p}_0)}$$

$$\text{and } k = \frac{2N\tilde{p}_0}{\tilde{\rho}_0^{\,2}} T'(\tilde{p}_0) + \frac{N\tilde{p}_0^{\,2}}{\tilde{\rho}_0^{\,2}} T''(\tilde{p}_0) + C_N^2 V''(\tilde{\rho}_0)$$

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$$\boldsymbol{Q} = \phi \sum_{i=1}^{N-1} \left(n_i + \frac{1}{2} \right) + \lambda$$

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- To choose l_i and calculate λ,
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- To compute ϕ,
- To choose n_i and calculate Q (do not forget ϕ),
- To resolve ET compact equations with this Q.

$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ \sqrt{C_N^2}\rho_0 p_0 = Q \\ Np_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \end{cases}$$

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The ET for systems of all identical particles

Example

Hamiltonian: $T(x) = Fx^\alpha$ $V(x) = \mathrm{sgn}(\beta)Gx^\beta$

$\Rightarrow$ Determination of $\tilde{\rho}_0$ and $\tilde{p}_0$:

$$\tilde{\rho}_0 = \left(\frac{N\alpha F\lambda^\alpha}{|\beta|G\sqrt{C_N}^{\alpha+2}}\right)^{1/(\alpha+\beta)} \quad \text{and} \quad \tilde{p}_0 = \frac{\lambda}{\sqrt{C_N}\tilde{\rho}_0}$$

Result is from [9] arXiv:2111.14744 (to appear in Few-Body Syst.)

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⇒ Computation of ϕ:

$$\phi = \sqrt{\alpha+\beta}$$

⇒ Final spectrum:

$$E = \text{sgn}(\beta)(\beta+\alpha) \left(\left(\frac{NF}{|\beta|}\right)^\beta \left(\frac{G}{\alpha}\right)^\alpha \left(\sqrt{C_N^2}\right)^{2\alpha-\alpha\beta} Q^{\alpha\beta} \right)^{1/(\alpha+\beta)}$$

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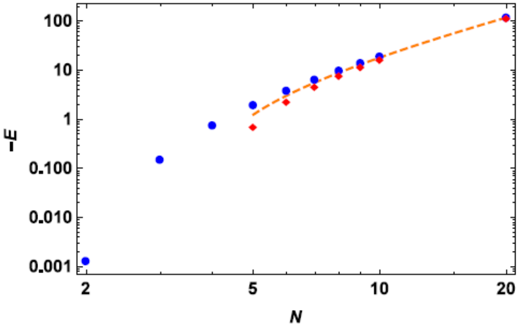

Figure: Biding energy for weakly-interacting bosons (gaussian interaction) with $d = 3$ - Exact results in circles, ET results in diamonds, $\phi = 1.82$ results in dashed line.

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

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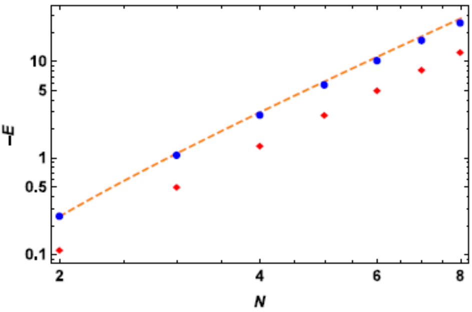

Figure: Biding energy for self-gravitating bosons (coulomb interaction) with $d = 3$ - Exact results in circles, ET results in diamonds, $\phi = 1$ results in dashed line.

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

The ET for systems of all identical particles

Tests

$n_1 + n_2$	$l_1 + l_2$	Exact	ET ($\phi = 2$)	ET ($\phi = \sqrt{2}$)
0	0	2.128	2.468	2.165
0	1	2.606	2.914	2.662
1	0	2.739	3.300	2.842
0	2	2.959	3.300	3.080
1	1	3.125	3.646	3.237
0	3	3.299	3.646	3.448
2	0	3.260	3.961	3.387
1	2	3.422	3.961	3.589
0	4	3.581	3.961	3.780
Δ			15%	3.8%

Table: Eigenmasses in GeV given by a model of light baryons ($D = 3$ and $N = 3$).

Results are from [5] Semay (2015) Few-Body Syst., **56**, 149

The ET for systems of all identical particles

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The ET and DOSM for systems of $N_a + 1$ particles

Why to generalize ?

The ET and DOSM for systems of $N_a + 1$ particles

Why to generalize ?

Hybrid baryons: three quarks + one constituent gluon

The ET and DOSM for systems of $N_a + 1$ particles

Why to generalize ?

Hybrid baryons: three quarks + one constituent gluon

- Large-N approach of QCD :
  - N quarks + one gluon

The ET and DOSM for systems of $N_a + 1$ particles

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[9] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

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the next system gives an approximation for its spectrum [9]:

$$\begin{cases} E=N_a T_a(\pi_0')+T_b(p_0)+C^2_{N_a} V_{aa}(\rho_0)+N_a V_{ab}(\rho_0') \\ T_a'(\pi_0')\frac{\pi_0^2}{\pi_0'}=C^2_{N_a} V_{aa}'(\rho_0)\rho_0+\frac{N_a-1}{2} V_{ab}'(\rho_0')\frac{\rho_0^2}{\rho_0'} \\ T_b'(p_0)p_0+\frac{1}{N_a} T_a'(\pi_0')\frac{p_0^2}{\pi_0'}=N_a V_{ab}'(\rho_0')\frac{r_0^2}{\rho_0'} \\ \pi_0\rho_0\sqrt{\frac{N_a-1}{2}}=Q(N_a) \\ p_0 r_0=Q(2) \end{cases}$$

- with ${\pi_0'}^2=\frac{\pi_0^2}{N_a}+\frac{p_0^2}{N_a^2}$ and ${\rho_0'}^2=\frac{N_a-1}{2N_a}\rho_0^2+r_0^2$
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Comparison with compact equations for systems of all identical particles:

$$\begin{cases} E = NT(p_0) + C_N^2 V(\rho_0) \\ N p_0 T'(p_0) = C_N^2 \rho_0 V'(\rho_0) \\ \sqrt{C_N^2}\rho_0 p_0 = Q \end{cases} \longleftrightarrow \begin{cases} E = N_a T_a(\pi_0') + T_b(p_0) + C_{N_a}^2 V_{aa}(\rho_0) + N_a V_{ab}(\rho_0') \\ T_a'(\pi_0')\frac{\pi_0^2}{\pi_0'} = C_{N_a}^2 V_{aa}'(\rho_0)\rho_0 + \frac{N_a-1}{2} V_{ab}'(\rho_0')\frac{\rho_0^2}{\rho_0'} \\ T_b'(p_0)p_0 + \frac{1}{N_a} T_a'(\pi_0')\frac{p_0^2}{\pi_0'} = N_a V_{ab}'(\rho_0')\frac{r_0^2}{\rho_0'} \\ \pi_0 \rho_0 \sqrt{\frac{N_a-1}{2}} = Q(N_a) \\ p_0 r_0 = Q(2) \end{cases}$$

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The ET and DOSM for systems of $N_a + 1$ particles

To apply the DOSM to the compact equations

Strategy [10]:

(1) To start with a classical purely orbital solution
(2) To start a radial perturbation (still classically)
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And after comparison of ET and DOSM, we get:

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$$\phi_{N_a} = \frac{\lambda_{N_a}}{B_{N_a}}\sqrt{\frac{2A''}{(N_a-1)m}} \quad \text{and} \quad \phi_b = \frac{\lambda_b}{B_b}\sqrt{\frac{B''}{m}}$$

where $B_{N_a} = T_a'(\tilde{\pi}_0{}')\frac{\tilde{\pi}_0{}^2}{\tilde{\pi}_0{}'}$ and $B_b = T_a'(\tilde{\pi}_0{}')\frac{\tilde{p}_0{}^2}{N_a\tilde{\pi}_0{}'} + T_b'(\tilde{p}_0)\tilde{p}_0$,

$m = \sqrt{\mu_a\mu_b}$,

$$\begin{cases} A'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a \\ B'' = \sqrt{\frac{\mu_a}{\mu_b}}k_b \end{cases} \quad \text{if } k_c = 0,$$

$$\begin{cases} A'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a - \frac{k_c}{2} \\ B'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a + \frac{k_c}{2} \end{cases} \quad \text{if } \epsilon = \frac{1}{k_c}\left(\sqrt{\frac{\mu_a}{\mu_b}}k_b - \sqrt{\frac{\mu_b}{\mu_a}}k_a\right) = 0,$$

$$\begin{cases} A'' = \sqrt{\frac{\mu_b}{\mu_a}}k_a - \frac{k_c}{2}\left(\frac{\epsilon}{|\epsilon|}\sqrt{1+\epsilon^2} - \epsilon\right) \\ B'' = \sqrt{\frac{\mu_a}{\mu_b}}k_b + \frac{k_c}{2}\left(\frac{\epsilon}{|\epsilon|}\sqrt{1+\epsilon^2} - \epsilon\right) \end{cases} \quad \text{if } \epsilon \neq 0,$$

$$\mu_a = \frac{\tilde{\pi}_0{}'}{T_a'(\tilde{\pi}_0{}')} \text{ and } \mu_b = \left(\frac{T_a'(\tilde{\pi}_0{}')}{N_a\tilde{\pi}_0{}'} + \frac{T_b'(\tilde{p}_0)}{\tilde{p}_0}\right)^{-1},$$

$$k_a = \frac{T_a''(\tilde{\pi}_0{}')\tilde{\pi}_0{}^4}{N_a\tilde{\rho}_0{}^2\tilde{\pi}_0{}'^{\,2}} + \frac{T_a'(\tilde{\pi}_0{}')\tilde{\pi}_0{}^2}{\tilde{\rho}_0{}^2}\left(\frac{3}{\tilde{\pi}_0{}'} - \frac{\tilde{\pi}_0{}^2}{N_a\tilde{\pi}_0{}'^{\,3}}\right) + C_{N_a}^2 V_{aa}''(\tilde{\rho}_0)$$
$$+ \frac{(N_a-1)^2\tilde{\rho}_0{}^2}{4N_a\tilde{\rho}_0{}'^{\,2}}V_{ab}''(\tilde{\rho}_0{}') + \frac{(N_a-1)}{2}\left(\frac{1}{\tilde{\rho}_0{}'} - \frac{(N_a-1)\tilde{\rho}_0{}^2}{2N_a\tilde{\rho}_0{}'^{\,3}}\right)V_{ab}'(\tilde{\rho}_0{}'),$$

$$k_b = \frac{T_a''(\tilde{\pi}_0{}')\tilde{p}_0{}^4}{N_a^3\tilde{r}_0{}^2\tilde{\pi}_0{}'^{\,2}} + \frac{T_b''(\tilde{p}_0)\tilde{p}_0{}^2}{\tilde{r}_0{}^2} + \frac{T_a'(\tilde{\pi}_0{}')\tilde{p}_0{}^2}{N_a\tilde{r}_0{}^2}\left(\frac{3}{\tilde{\pi}_0{}'} - \frac{\tilde{p}_0{}^2}{N_a^2\tilde{\pi}_0{}'^{\,3}}\right)$$
$$+ \frac{2T_b'(\tilde{p}_0)\tilde{p}_0}{\tilde{r}_0{}^2} + \frac{N_a\tilde{r}_0{}^2}{\tilde{\rho}_0{}'^{\,2}}V_{ab}''(\tilde{\rho}_0{}') + N_a\left(\frac{1}{\tilde{\rho}_0{}'} - \frac{\tilde{r}_0{}^2}{\tilde{\rho}_0{}'^{\,3}}\right)V_{ab}'(\tilde{\rho}_0{}'),$$

$$k_c = \frac{2\tilde{\pi}_0{}^2\tilde{p}_0{}^2}{N_a^2\tilde{\pi}_0{}'^{\,2}\tilde{\rho}_0\tilde{r}_0}\left(T_a''(\tilde{\pi}_0{}') - \frac{T_a'(\tilde{\pi}_0{}')}{\tilde{\pi}_0{}'}\right) + \frac{(N_a-1)\tilde{\rho}_0\tilde{r}_0}{\tilde{\rho}_0{}'^{\,2}}\left(V_{ab}''(\tilde{\rho}_0{}') - \frac{V_{ab}'(\tilde{\rho}_0{}')}{\tilde{\rho}_0{}'}\right).$$

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Tests

Test: $H = \sum_{i=1}^{3} |\vec{p}_i| + (\vec{r}_1 - \vec{r}_2)^2 + \kappa \sum_{i=1}^{2} (\vec{r}_i - \vec{r}_3)^2 \quad (D = 3)$

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- $\kappa=0.1,10$:

(n_a,n_b,l_a,l_b)	κ	Exact [8,9]	ET	$\Delta(\%)$	DOSM	$\Delta(\%)$
$(0,0,0,0)$	0.1	5.288	5.597	5.5	5.307	0.4
	10	14.506	15.352	5.8	14.699	1.3
$(0,0,1,1)$	0.1	7.515	7.868	4.7	7.625	1.5
	10	20.340	21.580	6.1	21.032	3.4
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[11] Semay, Sicorello (2018) Few-Body Syst., **59**, 119

[10] Semay, Cimino, Willemyns (2020) Few-Body Syst., **61**, 19

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